

EXPLICIT FORMULAS INVOLVING q -EULER NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, we deal with q -Euler numbers and q -Bernoulli numbers. We derive some interesting relations for q -Euler numbers and polynomials by using their generating function and derivative operator. Also, we show between the q -Euler numbers and q -Bernoulli numbers via the p -adic q -integral in the p -adic integer ring.

1. PRELIMINARIES

Imagine that p be a fixed odd prime number. Throughout this paper we use the following notations, by \mathbb{Z}_p denotes the ring of p -adic rational integers, \mathbb{Q} denotes the field of rational numbers, \mathbb{Q}_p denotes the field of p -adic rational numbers, and \mathbb{C}_p denotes the completion of algebraic closure of \mathbb{Q}_p . Let \mathbb{N} be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

The p -adic absolute value is defined by

$$|p|_p = \frac{1}{p}.$$

In this paper we assume $|q - 1|_p < 1$ as an indeterminate.

$[x]_q$ is a q -extension of x which is defined by

$$[x]_q = \frac{1 - q^x}{1 - q},$$

we note that $\lim_{q \rightarrow 1} [x]_q = x$ (see[1-12]).

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

has a limit $f'(a)$ as $(x, y) \rightarrow (a, a)$ and denote this by $f \in UD(\mathbb{Z}_p)$.

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable function on \mathbb{Z}_p . For $f \in UD(\mathbb{Z}_p)$, let us start with the expressions

$$\frac{1}{[p^N]_q} \sum_{0 \leq \xi < p^N} f(\xi) q^\xi = \sum_{0 \leq \xi < p^N} f(\xi) \mu_q(\xi + p^N \mathbb{Z}_p),$$

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represents p -adic q -analogue of Riemann sums for f . The integral of f on \mathbb{Z}_p will be defined as the limit ($N \rightarrow \infty$) of these sums, when it exists. The p -adic q -integral of function $f \in UD(\mathbb{Z}_p)$ is defined by T. Kim

$$(1.1) \quad I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_q(\xi) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{\xi=0}^{p^N-1} f(\xi) q^\xi$$

The bosonic integral is considered as a bosonic limit $q \rightarrow 1$, $I_1(f) = \lim_{q \rightarrow 1} I_q(f)$. Similarly, the fermionic p -adic integral on \mathbb{Z}_p is introduced by T. Kim as follows:

$$(1.2) \quad I_{-q}(f) = \lim_{q \rightarrow -q} I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_{-q}(\xi)$$

(for more details, see [9-12]).

In [6], the q -Euler polynomials with weight 0 are introduced as

$$(1.3) \quad \tilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y)$$

From (1.3), we have

$$\tilde{E}_{n,q}(x) = \sum_{l=0}^n \binom{n}{l} x^l \tilde{E}_{n-l,q}$$

where $\tilde{E}_{n,q}(0) = \tilde{E}_{n,q}$ are called q -Euler numbers with weight 0. Then, q -Euler numbers are defined as

$$q \left(\tilde{E}_q + 1 \right)^n + \tilde{E}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention about replacing $\left(\tilde{E}_q \right)^n$ by $\tilde{E}_{n,q}$ is used.

Similarly, the q -Bernoulli polynomials and numbers with weight 0 are defined, respectively

$$\begin{aligned} \tilde{B}_{n,q}(x) &= \lim_{n \rightarrow \infty} \frac{1}{[p^n]_q} \sum_{y=0}^{p^n-1} (x+y)^n q^y \\ &= \int_{\mathbb{Z}_p} (x+y)^n d\mu_q(y) \end{aligned}$$

and

$$\tilde{B}_{n,q} = \int_{\mathbb{Z}_p} y^n d\mu_q(y)$$

(for more informations, see [4]).

We, by using Kim's et al. method in [2], will investigate some interesting identities on the q -Euler numbers and polynomials from their generating function and derivative operator. Consequently, we derive some properties on q -Euler numbers and polynomials and q -Bernoulli numbers and polynomials by using q -Volkenborn integral and fermionic p -adic q -integral on \mathbb{Z}_p .

2. ON KIM'S q -EULER NUMBERS AND POLYNOMIALS

Let us consider Kim's q -Euler polynomials as follows:

$$(2.1) \quad F_x^q = F_x^q(t) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!}.$$

Here x is a fixed parameter. Thus, by expression of (2.1), we can readily see the following

$$(2.2) \quad qe^t F_x^q + F_x^q = [2]_q e^{xt}.$$

Last from equality, taking derivative operator D as $D = \frac{d}{dt}$ on the both sides of (2.2). Then, we easily see that

$$(2.3) \quad qe^t (D + I)^k F_x^q + D^k F_x^q = [2]_q x^k e^{xt}$$

where $k \in \mathbb{N}^*$ and I is identity operator. By multiplying e^{-t} on both sides of (2.3), we get

$$(2.4) \quad q (D + I)^k F_x^q + e^{-t} D^k F_x^q = [2]_q x^k e^{(x-1)t}$$

Let us take derivative operator D^m ($m \in \mathbb{N}$) on both sides of (2.4). Then we get

$$(2.5) \quad qe^t D^m (D + I)^k F_x^q + D^k (D - I)^m F_x^q = [2]_q x^k (x - 1)^m e^{xt}$$

Let $G[0]$ (not $G(0)$) be the constant term in a Laurent series of $G(t)$. Then, from (2.5), we get

$$(2.6) \quad \sum_{j=0}^k \binom{k}{j} (qe^t D^{k+m-j} F_x^q(t)) [0] + \sum_{j=0}^m \binom{m}{j} (-1)^j (D^{k+m-j} F_x^q(t)) [0] = [2]_q x^k (x - 1)^m$$

By (2.1), we see

$$(2.7) \quad (D^N F_x^q(t)) [0] = \tilde{E}_{N,q}(x) \text{ and } (e^t D^N F_x^q(t)) [0] = \tilde{E}_{N,q}(x)$$

By expressions of (2.6) and (2.7), we see that

$$(2.8) \quad \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \tilde{E}_{k+m-j,q}(x) = [2]_q x^k (x - 1)^m.$$

From (2.1), we note that

$$(2.9) \quad \frac{d}{dx} (\tilde{E}_{n,q}(x)) = n \sum_{l=0}^{n-1} \binom{n-1}{l} \tilde{E}_{l,q} x^{n-1-l} = n \tilde{E}_{n-1,q}(x)$$

By (2.9), we easily see,

$$(2.10) \quad \int_0^1 \tilde{E}_{n,q}(x) dx = \frac{\tilde{E}_{n+1,q}(1) - \tilde{E}_{n+1,q}}{n+1} = -\frac{[2]_{q^{-1}}}{n+1} \tilde{E}_{n+1,q}$$

Now, let us consider definition of integral from 0 to 1 in (2.8), then we have

$$\begin{aligned}
 (2.11) \quad & -[2]_{q^{-1}} \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \frac{\tilde{E}_{k+m-j+1,q}}{k+m-j+1} \\
 & = [2]_q (-1)^m B(k+1, m+1) \\
 & = [2]_q (-1)^m \frac{\Gamma(k+1) \Gamma(m+1)}{\Gamma(k+m+2)}
 \end{aligned}$$

where $B(m, n)$ is beta function which is defined by

$$\begin{aligned}
 B(m, n) & = \int_0^1 x^{m-1} (1-x)^{n-1} dx \\
 & = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, \quad m > 0 \text{ and } n > 0.
 \end{aligned}$$

As a result, we obtain the following theorem

Theorem 1. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned}
 & \sum_{j=1}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \frac{\tilde{E}_{k+m-j+1,q}}{k+m-j+1} \\
 & = q \frac{(-1)^{m+1}}{(k+m+1) \binom{k+m}{k}} - [2]_q \frac{\tilde{E}_{k+m+1,q}}{k+m+1}.
 \end{aligned}$$

Substituting $m = k+1$ into Theorem 1, we readily get

$$\begin{aligned}
 & \sum_{j=1}^{k+1} \left[q \binom{k}{j} + (-1)^j \binom{k+1}{j} \right] \frac{\tilde{E}_{2k+2-j,q}}{2k+2-j} \\
 & = q \frac{(-1)^k}{(2k+2) \binom{2k+1}{k}} - [2]_q \frac{\tilde{E}_{2k+2,q}}{2k+2}.
 \end{aligned}$$

By (2.1), it follows that

$$\begin{aligned}
 & \sum_{j=0}^{\max\{k,m\}} (k+m-j) \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \tilde{E}_{k+m-j-1,q}(x) \\
 & = [2]_q x^{k-1} (x-1)^{m-1} ((k+m)x - k).
 \end{aligned}$$

Let $m = k$ in (2.1), we see that

$$\sum_{j=0}^k \left[q \binom{k}{j} + (-1)^j \binom{k}{j} \right] \tilde{E}_{2k-j,q}(x) = [2]_q x^k (x-1)^k$$

Last from equality, we discover the following

$$\begin{aligned}
 (2.12) \quad & [2]_q \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \tilde{E}_{2k-2j,q}(x) + (q-1) \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \tilde{E}_{2k-2j-1,q}(x) = [2]_q x^k (x-1)^k.
 \end{aligned}$$

Here $[.]$ is Gauss' symbol. Then, taking integral from 0 to 1 both sides of last equality, we get

$$\begin{aligned}
& -[2]_{q^{-1}} [2]_q \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \frac{\tilde{E}_{2k-2j+1,q}}{2k-2j+1} + [2]_{q^{-1}} (1-q) \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \frac{\tilde{E}_{2k-2j,q}}{2k-2j} \\
&= [2]_q (-1)^k B(k+1, k+1) \\
&= \frac{[2]_q (-1)^k}{(2k+1) \binom{2k}{k}}.
\end{aligned}$$

Consequently, we derive the following theorem

Theorem 2. *The following identity*

$$\begin{aligned}
(2.13) \quad & [2]_q \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \frac{\tilde{E}_{2k-2j+1,q}}{2k-2j+1} + (q-1) \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \frac{\tilde{E}_{2k-2j,q}}{2k-2j} \\
&= \frac{q(-1)^{k+1}}{(2k+1) \binom{2k}{k}}.
\end{aligned}$$

is true.

In view of (2.1) and (2.12), we discover the following applications:

$$\begin{aligned}
(2.14) = & \sum_{j=0}^{k+1} \left[q \binom{k}{j} + (-1)^j \binom{k+1}{j} \right] \tilde{E}_{2k+1-j,q}(x) \\
&= [2]_q \tilde{E}_{2k+1,q}(x) + \sum_{j=1}^{\left[\frac{k+1}{2}\right]} \left[q \binom{k}{2j} + \binom{k}{2j} + \binom{k}{2j-1} \right] \tilde{E}_{2k+1-2j,q}(x) \\
&\quad + \sum_{j=0}^{\left[\frac{k+1}{2}\right]} \left[q \binom{k}{2j+1} - \binom{k}{2j+1} - \binom{k}{2j} \right] \tilde{E}_{2k-2j,q}(x) \\
&= - \left[\sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \tilde{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \tilde{E}_{2k-2j+1}(x) \right] \\
&\quad + [2]_q \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \tilde{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{\left[\frac{k}{2}\right]} \binom{k}{2j-1} \tilde{E}_{2k+1-2j,q}(x) \\
&\quad + (q-1) \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \tilde{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \tilde{E}_{2k-2j+1}(x)
\end{aligned}$$

By expressions (2.12) and (2.14), we have the following Theorem

Theorem 3. For $k \in \mathbb{N}$, we have

$$\begin{aligned}
 (2.15) \quad & [2]_q \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \tilde{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{\left[\frac{k}{2}\right]} \binom{k}{2j-1} \tilde{E}_{2k+1-2j,q}(x) \\
 & + (q-1) \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \left[\tilde{E}_{2k-2j,q}(x) + \frac{1}{1+q} \tilde{E}_{2k-2j+1}(x) \right] \\
 & = x^k (x-1)^k ([2]_q x - q)
 \end{aligned}$$

3. p -adic integral on \mathbb{Z}_p associated with Kim's q -Euler polynomials

In this section, we consider Kim's q -Euler polynomials by means of p -adic q -integral on \mathbb{Z}_p . Now we start with the following assertion.

Let $m, k \in \mathbb{N}$, Then by (2.8),

$$\begin{aligned}
 I_1 &= [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu_{-q}(x) \\
 &= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} d\mu_{-q}(x) \\
 &= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}
 \end{aligned}$$

On the other hand, right hand side of (2.8),

$$\begin{aligned}
 I_2 &= \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\
 &= \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q}
 \end{aligned}$$

Equating I_1 and I_2 , we get the following theorem

Theorem 4. For $m, k \in \mathbb{N}$, we have

$$\begin{aligned}
 & \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q} \\
 &= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}.
 \end{aligned}$$

Let us take fermionic p -adic q -inetgral on \mathbb{Z}_p left hand side of (2.15), we get

$$\begin{aligned}
 I_3 &= \int_{\mathbb{Z}_p} x^k (x-1)^k ([2]_q x - q) d\mu_{-q}(x) \\
 &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_{-q}(x) - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-q}(x) \\
 &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l+1,q} - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l,q}
 \end{aligned}$$

In other word, we consider right hand side of (2.15) as follows:

$$\begin{aligned}
I_4 &= [2]_q \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\
&\quad + \sum_{j=1}^{\left[\frac{k}{2}\right]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\
&\quad + \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \left[\begin{array}{l} (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\ + \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \end{array} \right] \\
&= [2]_q \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=1}^{\left[\frac{k}{2}\right]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \left[\begin{array}{l} (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{E}_{l,q} \\ + \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \tilde{E}_{l,q} \end{array} \right]
\end{aligned}$$

Equating I_3 and I_4 , we get the following theorem

Theorem 5. For $k \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left[[2]_q \tilde{E}_{k+l+1,q} - q \tilde{E}_{k+l,q} \right] \\
&= [2]_q \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=1}^{\left[\frac{k}{2}\right]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \left\{ \begin{array}{l} (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{E}_{l,q} \\ + \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \tilde{E}_{l,q} \end{array} \right\}
\end{aligned}$$

Now, we consider (2.8) and (2.1) by means of q -Volkenborn integral. Then, by (2.8), we see

$$\begin{aligned}
&[2]_q \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu_q(x) \\
&= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} d\mu_q(x) \\
&= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{B}_{l+k,q}
\end{aligned}$$

On the other hand,

$$\begin{aligned} & \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ &= \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{B}_{l,q} \end{aligned}$$

Therefore, we get the following theorem

Theorem 6. *For $m, k \in \mathbb{N}$, we have*

$$\begin{aligned} & [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{B}_{l+k,q} \\ &= \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{B}_{l,q} \end{aligned}$$

By using fermionic p -adic q -integral on \mathbb{Z}_p left hand side of (2.15), we get

$$\begin{aligned} I_5 &= [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^k ([2]x-q) d\mu_q(x) \\ &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_q(x) - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x) \\ &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l+1,q} - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l,q} \end{aligned}$$

Also, we consider right hand side of (2.15) as follows:

$$\begin{aligned} I_6 &= [2]_q \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ &\quad + \sum_{j=1}^{\left[\frac{k}{2}\right]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ &\quad + \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \left[\begin{array}{l} (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ + \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \end{array} \right] \\ &= [2]_q \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \\ &\quad + \sum_{j=1}^{\left[\frac{k}{2}\right]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \\ &\quad + \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \left[\begin{array}{l} (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{B}_{l,q} \\ + \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \tilde{B}_{l,q} \end{array} \right] \end{aligned}$$

Equating I_5 and I_6 , we get the following Corollary

Corollary 1. For $k \in \mathbb{N}$, we get

$$\begin{aligned}
 & \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left[[2]_q \tilde{B}_{k+l+1,q} - q \tilde{B}_{k+l,q} \right] \\
 = & [2]_q \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \\
 & + \sum_{j=1}^{\left[\frac{k}{2}\right]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \\
 & + \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k}{2j+1} \left\{ \begin{array}{l} (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{B}_{l,q} \\ + \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \tilde{B}_{l,q} \end{array} \right\}
 \end{aligned}$$

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